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REPRESENTATION THEORETIC INVESTIGATION ON THE NUMBER OF CONJUGACY CLASSES OF TWO FAMILIES OF GROUPS

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ABSTRACT
In this paper, we will solve the problem of counting the number of conjugacy classes of groups of order pq with p, q distinct primes and the number of conjugacy classes of groups of order p^3, where p is prime. In general, Sylow Theory together with class equation for groups is used for solving these problems, but instead of it we have used representation theory to solve these problems.

MSC 2020: 20C15, 20E45
KEYWORDS: Irreducible representation, equivalent representation, Character, Regular representation, direct sum of representations.

1. INTRODUCTION
Representation theory of groups is used to model abstract groups with concrete structures like subgroups of matrices by identifying group elements with $n \times n$ invertible matrices and group operations with matrix multiplication. Representation theory helps us to reduce problems in abstract algebra to problems in linear algebra, and linear algebra is easier than abstract algebra.

Irreducible representations of groups are essential in Representation Theory, because Maschke's theorem tells that every representation of a finite group is a direct sum of its irreducible representations, i.e. irreducible representations are the building blocks of every representation of a finite group. If we can count the number of irreducible representations of each degree, then we can completely understand the structure of the given representation.

In [1], G. Frobenius proved in the year 1896 that "the degree of irreducible representations of a finite group divides the order of the group". J. Schur improved this theorem in 1904. J. Schur in [2] proved that "the degree of irreducible representation of a finite group G divides the index of the center of G in a group G". In 1951 , N. Itô [3] proved that "the degree of an irreducible representation of a finite group G divides the index of its maximal abelian normal subgroups in G".

Recently, Lili Li [4] studied the number of conjugacy classes of nonnormal subgroups of a finite group G. In [5] & [6], Gustavo A. Fernandez-Alcober and Leire Legarreta studied the lower bounds on the number of conjugacy classes of non-normal subgroups and normalizer subgroups. In [7], Bilal Al-Hasanat & et.al investigated an upper bound for the number of conjugacy classes of non-abelian nilpotent groups.

In [8, p. 77], as a corollary to the dimension theorem, it is stated that any group G of order pq is abelian, where p, q are primes with $p < q$ and $q \not\equiv 1 \mod p$. We investigated the situation when the group is of order pq, where p, q are primes with $p < q$ and $q \equiv 1 \mod p$.

In general, in the representation theory of finite groups, the information about the number of conjugacy classes of groups is used to know the number of inequivalent irreducible representations of groups and then one constructs that many irreducible representations of that group. In this paper, we use the information about the number of inequivalent irreducible representations of groups to know the number of conjugacy classes of groups of order pq, where p, q are primes with $p < q$ *and* $q \not\equiv 1 \bmod p$ and the number of conjugacy classes of groups of order p^3, where p is prime.

2. MATERIALS AND METHODS

Definitions and Examples

1. Representation: Suppose G is a group and V is a finite dimensional non-zero vector space over the field of complex numbers $\mathbb{C}$. A representation of a group G is a group homomorphism $\rho: G \to GL(V)$ and it is denoted by (ρ, V). We will denote $\rho(g)$ by ρ_g.
If $dim(V) = n$, then we can identify $GL(V)$ with $GL_n(\mathbb{C})$.
Trivial representation: A trivial representation of a group G is a group homomorphism
$\rho: G \to \mathbb{C}^*$ such that $\rho(g) = 1 \; \forall \; g \in G$.

2. Subrepresentation: Suppose $\rho: G \to GL(V)$ is a representation. A subspace W of V is called a subrepresentation if it is G-invariant, i.e., $\rho_g w \in W$ for all $g \in G$ and $w \in W$.
3. Direct sum of representations: Suppose $\rho : G \to GL(V)$ and $\phi: G \to GL(W)$ are representations. Then their direct sum

$$\rho \oplus \phi : G \to GL(V \oplus W)$$

is given by
$(\rho \oplus \phi)_g(v, w) = (\rho_g v, \phi_g w) \;\; \forall g \in G$.

4. Irreducible representation: A nonzero representation V is said to be an irreducible representation if it has only two subrepresentations, namely, $\{0\}$ and V itself.

5. Reducible Representation: If a representation (ρ, V) has a proper nonzero subrepresentation, then it is said to be a reducible representation.

6. Decomposable Representation: A representation (ρ, V) is decomposable if $V = V_1 \oplus V_2$, where V_1 and V_2 are proper nonzero subrepresentations.
Otherwise, the representation is said to be indecomposable.

***7. Equivalent Representations*:** Suppose V and W are vector spaces over a field K, then the representations $\rho : G \to GL(V)$ and $\phi: G \to GL(W)$ are said to be equivalent if ∃ a vector space isomorphism $T : V \to W$ so that for all g in G, we have $T \circ \rho(g) \circ T^{-1} = \phi(g)$.

8. Completely reducible representation: A representation (ρ, V) is said to be completely reducible if it is a direct sum of irreducible representations.

9. Character: **(** [9, p. 30]**):** "Suppose $X : G \to GL_n(\mathbb{C})$ is a matrix representation of a group G. Then the character χ of X is the function $\chi: G \to \mathbb{C}$ given by $\chi(g) = \text{trace}(X(g))$, for all $g \in G$".

10. Regular representation: **(** [8, p. 42] **):** "Suppose G is a finite group. The regular representation of G is a homomorphism $L: G \to GL(\mathbb{C}G)$ defined by $Lg(\sum c_h h) = \sum c_h gh = \sum c_{g^{-1}x} \, x$ for $g \in G$ and h, x runs over all elements of G".

Now we are stating some results in the representation theory of groups required to understand our research.

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Required Theorems

Theorem 1 ([8, pp. 43-44]**) :** Suppose L is a regular representation of G . Then the representation L is equivalent to $d_1\,\phi^{(1)} \oplus d_2\,\phi^{(2)} \oplus \ldots \oplus d_s\,\phi^{(s)}$ where $\{\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(s)}\}$ is a complete list of inequivalent irreducible representations of finite group G and $d_i = deg(\phi^{(i)})$ and the formula $|G| = d_1^2 + d_2^2 + \cdots + d_s^2$ is satisfied.

Theorem 2 ([8, p. 45]**) :** The number of inequivalent irreducible representations of a finite group G is equal to the number of conjugacy classes of G.

Theorem 3 ([8, p. 45]**):** A finite group G is an abelian iff the number of inequivalent irreducible representations of G is equal to the order of group G.

Theorem 4 (Dimension Theorem ([8, p. 76]**):** Suppose ϕ is an irreducible representation of a group G of degree d. Then d divides the order of group G.

Theorem 5 ([8, p. 77]**):** Suppose G is a finite group. Then the number of degree one representations of group G divides the order of the group G.

Theorem 6 ([8, p. 77]**):** Suppose p is a prime and the order of group G is p^2. Then G is abelian.

Proof: Suppose $d_1, d_2, \ldots, d_s$ are the degrees of the irreducible representations of group G. Then by Theorem 1, we get

$|G| = p^2 = d_1^2 + d_2^2 + \cdots + d_s^2$ with p prime . ... (1)

Since by Theorem 4, the degrees of the irreducible representations of the group divide the order of the group, d_i can be $1, p$, or p^2. We know that every group has the trivial representation of degree 1, therefore all d_i's must be 1, otherwise we get a contradiction to equation (1). Since all irreducible representations of G have degree 1, G is abelian by Theorem 3.

Theorem 7 ([8, p. 77]): Let p, q be primes with $p < q$ and $q \not\equiv 1 \bmod p$. Then any group G of order pq is abelian.

Let G be a group of order pq with p , q distinct primes and $p < q$ and q is not congruent to 1 modulo p. Suppose $d_1, d_2, \ldots, d_s$ are the degrees of the irreducible representations of the group G.Then by Theorem 1, we get

$|G| = pq = d_1^2 + d_2^2 + \cdots + d_s^2$ with p, q distinct primes and $p < q$ … (2)

Since by Theorem 4, the degrees of the irreducible representations of the group divide the order of the group and $p < q$, d_i can be 1 or p for all i. Let n denote the number of degree p representations of G and let m denote the number of degree 1 representations of G. Then,

$$pq = m + np^2 \quad \text{... (3)}$$

Every group has a trivial representation, therefore $m \geq 1$. From equation (3), we get $p \mid m$. By Theorem 5, we know that the number of degree 1 representations of G divides $|G|$. i.e., $m \mid pq$. Therefore , $m = p$ or $m = pq$. If $m = p$, then $q = 1 + np$ contradicts that q is not congruent to 1 modulo p. Therefore, $m = pq$ and hence all irreducible representations of the group G have degree one. Therefore, G is abelian by theorem 3.

3. RESULTS AND DISCUSSION

Result 1. Any nonabelian group of order pq with p , q distinct primes and $p < q, q \equiv 1 mod\, p$ has $p + (q - 1)/p$ conjugacy classes.

Proof:

Let G be a nonabelian group of order pq with p , q distinct primes and $p < q, q \equiv 1 mod\, p$. Suppose $d_1, d_2, \ldots, d_s$ are the degrees of the irreducible representations of group G. Then by Theorem 1,

$|G| = pq = d_1^2 + d_2^2 + \cdots + d_s^2$ ... (4)

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Since by Theorem 4, the degrees of the irreducible representations of the group divide the order of the group and $p < q$, d_i can be 1 or p for all i. Let n denote the number of degree p irreducible representations of G and let m denote the number of degree 1 irreducible representations of G. Then,

$$pq = m + np^2 \quad \text{... (5)}$$

Every group has a trivial representation, therefore $m \geq 1$. From equation (5), we get $p \mid m$. By Theorem 5, we know that the number of degree 1 representations of G divides $|G|$. i.e. $m \mid pq$. Therefore , $m = p$ or $m = pq$. If $m = pq$, then all irreducible representations of the group G have degree one and hence, G is abelian by Theorem 3. This leads to a contradiction since G is a nonabelian group.

Therefore, $m = p$. i.e., there are p number of degree 1 inequivalent irreducible representations of G. By putting this in equation (5), we get
$pq = p + np^2$,
$q = 1 + np$,
$n = (q - 1)/p$,
i.e., there are $(q - 1)/p$ number of degree p inequivalent irreducible representations of group G.
Therefore, the total number of inequivalent irreducible representations of group G is equal to
$p + (q - 1)/p$. By Theorem 2, we get the number of conjugacy classes of group G is equal to
$p + (q - 1)/p$.

Result 2. Any nonabelian group of order p^3, where p is prime, has $p^2 + (p - 1)$ conjugacy classes.

Proof: Let G be a nonabelian group of order p^3 where p is prime. Suppose $d_1, d_2, \ldots, d_s$ are the degrees of the irreducible representations of group G. Then by Theorem 1,

$$|G| = p^3 = d_1^2 + d_2^2 + \cdots + d_s^2 \quad \text{... (6)}$$

Since by Theorem 4, the degrees of the irreducible representations of the group divide the order of group, d_i can be 1 or p for all i. Let n denote the number of degree p irreducible representations of G and let m denote the number of degree 1 irreducible representations of G. Then,

$$p^3 = m + np^2 \quad \text{... (7)}$$

Every group has a trivial representation, therefore $m \geq 1$. From equation (7), we get $p^2 \mid m$. By Theorem 5, we know that the number of degree 1 representations of G divides $|G|$. i.e. $m \mid p^3$. Therefore , $m = p^2$ or $m = p^3$. If $m = p^3$, then all irreducible representations of the group G have degree one and hence, G is abelian by Theorem 3. This leads to a contradiction since G is a nonabelian group.

Therefore, $m = p^2$. i.e., there are p^2 number of degree 1 irreducible representations of G. By putting this in equation (7), we get

$$p^3 = p^2 + np^2$$
$$p = 1 + n$$
$$n = (p - 1)$$

i.e., there are $(p - 1)$ number of degree p irreducible representations of group G. Therefore, the total number of inequivalent irreducible representations of group G is equal to
$p^2 + p - 1$. By Theorem 2, we get the number of conjugacy classes of group G is equal to
$p^2 + p - 1$.

4. CONCLUSION

Our first result shows that any nonabelian group of order pq with p, q distinct primes and $p < q, q \equiv 1 \; mod \; p$ has $p + [(q - 1) \div p]$ conjugacy classes.

Our second result shows that any nonabelian group of order p^3, where p is prime, has $p^2 + p - 1$ conjugacy classes.

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REFERENCES

[1] G. Frobenius, Über Die Primfaktoren Der Gruppendeterminante, Sitz.Berlin. 1343, 1896.
[2] J. Schur, "Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen.," Journal für die reine und angewandte Mathematik, vol. 127, pp. 20-50, 1904.
[3] N. Itô, "On the Degrees of Irreducible Representations of a Finite Group," Nagoya Mathematical Journal, vol. 3, pp. 5-6, 1951.
[4] L. Li, "The number of conjugacy classes of nonnormal subgroups of finite p-groups (III)," Journal of Algebra and Its Applications, vol. 20, 2021.
[5] G. A. Fernandez-Alcober and L. Legarreta, "Counting Conjugacy Classes of Subgroups in Finite p-Groups, I," in Ischia Group Theory 2006, 2007.
[6] M. E. D. Marco, G. A. Fernandez-Alcober and L. Legarreta, "Counting Conjugacy Classes of Subgroups in Finite p-Groups, II," in Ischia Group Theory 2006, 2007.
[7] B. Al-Hasanat, A. Al-Dababseh, E.Al-Sarairah, S. Alobiady and M. B. Alhasanat, "An Upper Bound to the Number of Conjugacy Classes of Non-Abelian Nilpotent Groups," Journal of Mathematics and Statitics, vol. 13, no. 2, pp. 139-142, 2017.
[8] B. Steinberg, Representation Theory of Finite Groups: An Introductory Approach, Springer Science + Business Media, LLC, 2012.
[9] B. E. Sagan, The Symmetric Group Representations, Combinatorial Algorithms, And Symmetric Functions, Springer Science + Business Media, LLC, 2001.